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LETTER TO THE EDITOR

On the magnetic perturbation of the Ising model on the sphere

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Abstract

In this letter we will extend the analysis given by Zamolodchikov for the scaling Yang–Lee model on the sphere to the Ising model in a magnetic field. A numerical study of the partition function and of the vacuum expectation values is done by using the truncated conformal space approach. Our results strongly suggest that the partition function is an entire function of the coupling constant.

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1. Introduction

Quantum field theories on curved (fixed) background have attracted much attention and have been explored for a long time [1]. The main reason for studying them is their importance as a first step towards the understanding of quantum gravity. In this letter we will consider 2D conformal field theories (CFT) and their perturbations on a spherical background. Apart from their relevance in order to understand 2D quantum gravity, the theories on the sphere can be considered as infrared regularizations of the corresponding theories defined on the infinite plane; these latter can in general be recovered in the limit of infinite radius. This natural cut-off avoids the presence of infrared divergences in the perturbative theory and various physical quantities are analytic in the coupling constant λ at $\lambda = 0$ (see also footnote 5).

In a recent paper Zamolodchikov [2] proposed a novel way to study the partition function of the scaling Yang–Lee model on the sphere. Motivated by a numerical analysis he suggested that such partition function is an entire function of the coupling constant. Assuming that this conjecture is true, one can get useful information about the large R limit (R being the radius of the sphere) from the knowledge of the first few terms of the expansion coming from conformal perturbation theory.

It is interesting to test this approach considering other relevant perturbations of minimal models, in particular we will study the Ising model perturbed by a magnetic field. The former conjecture has been confirmed analytically [2] in the case of the thermal perturbation of the Ising model resorting to the fact that it is equivalent to a free massive Majorana fermion. The present case is less trivial (it is not a free theory) and requires to be investigated with the same numerical methods employed in [2]. In particular we address two main issues: the asymptotic behaviour of the vacuum expectation values (VEV) on the sphere; the asymptotic location of the zeroes of the partition function.

This letter is organized as follows. In section 2 we will briefly introduce CFT on the sphere and the perturbed theory. In section 3 we will describe the numerical results and finally in section 4 we will give our conclusions.

2. Ising model in a magnetic field

In the recent past, conformal field theories (and in particular minimal models) have been studied on a general Riemann surface [3]. In the following we will deal with models defined on a sphere³, the simplest non-trivial example of curved geometry. In particular, we will consider the first model of the minimal unitary series, i.e. the Ising model. It is characterized by a set of primary fields ϕ_i (1, σ and ϵ) which transform in the following way:

$$\delta\phi_i(x) = -\Delta_i\phi_i(x)\delta\chi(x) \quad g_{ab} = e^\chi\delta_{ab} \quad (1)$$

under a Weyl transformation

$$\delta g_{ab}(x) = g_{ab}(x)\delta\chi(x) \quad (2)$$

where

$$e^{\chi(x)} = \frac{4R^2}{1+z\bar{z}} \quad (3)$$

is the Weil factor of the metric, Δ_i are the conformal weights of primary fields, given respectively by 0, 1/16 and 1/2.

The trace of the energy–momentum tensor in CFTs defined upon non-trivial geometric backgrounds plays a central role. In fact, it gives a quantitative characterization of the effect of a change in the geometry. Its explicit form for the case of the sphere (with radius R) is given by

$$\theta(x) = -\frac{c}{12}\hat{R} \quad (4)$$

where $\hat{R} = \frac{2}{R^2}$ is the scalar curvature of the sphere and $c = 1/2$ is the central charge.

By equation (4) and by the definition of $\theta(x)$ in terms of the partition function $Z_{\text{CFT}}(R)$ one gets the relation

$$Z_{\text{CFT}}(R) = R^{c/3}Z_0 \quad (5)$$

where $Z_0 = Z_{\text{CFT}}(1)$.

As shown in [2], the action of a CFT perturbed by a relevant operator is given by

$$\mathcal{S}_\lambda = \mathcal{S}_{\text{CFT}} + \frac{\lambda}{2\pi} \int \sigma(x) e^{\chi(x)} d^2x. \quad (6)$$

\mathcal{S}_{CFT} is the action of the Ising model on the sphere and $\sigma(x)$ is the perturbing operator. The partition function $Z_\lambda(R)$ can be calculated in the regimes of both small and large values of R .

³ We assume that there are no conical singularities and the metric is smooth. In the presence of conical singularities there are metric-dependent terms in the partition function [5].

In the first case, expanding in λ and defining

$$h = \lambda(2R)^{2-2\Delta_\sigma} \tag{7}$$

and

$$z(h) = \frac{Z_\lambda(R)}{Z_0 R^{c/3}} \tag{8}$$

one gets

$$z(h) = \sum_{n=0}^{\infty} (-h)^n z_n \tag{9}$$

where $z_0 = 1$ and

$$z_n = \frac{\pi}{(2\pi)^n n!} \int \langle \sigma(0) \cdots \sigma(y_n) \rangle \prod_{i=2}^n \frac{d^2 y_i}{(1 + y_i \bar{y}_i)^{2-2\Delta_\sigma}} \tag{10}$$

and the correlators are calculated in the conformal theory on the plane. By using the fusion rules of the Ising model it is easy to show that only even coefficients are different from zero.

In [2], it is conjectured that this series is absolutely convergent and defines an entire function of h .

A large R expansion can be obtained by using the formula

$$\delta \langle X \rangle = -\frac{1}{4\pi} \int \langle \theta(x) X \rangle e^{\chi(x)} \delta \chi(x) d^2 x \tag{11}$$

which gives the variation of $\langle X \rangle$ in terms of insertions of $\theta(x)$. By applying this formula to $\theta(0)$ one gets

$$\langle \theta(0) \rangle_{\text{sphere}} \sim 4\pi \mathcal{E}_{\text{vac}} + \frac{b_1}{R^4} + \frac{2b_2}{R^6} + \dots \tag{12}$$

where A is defined in terms of vacuum energy in flat space:

$$\mathcal{E}_{\text{vac}} = -A\lambda^{1/(1-\Delta_\sigma)} \tag{13}$$

and the coefficients b_i can be expressed in terms of integrals of higher flat correlation functions of θ .

The derivative of the partition function with respect to R is given by

$$\frac{d \log Z_\lambda(R)}{dR^2} = -\langle \theta \rangle \tag{14}$$

and combining (12) and (14) it follows that

$$\log \frac{Z_\lambda(R)}{Z_0} \sim -4\pi R^2 \mathcal{E}_{\text{vac}} + \log(z_\infty) + \frac{b_1}{R^2} + \frac{b_2}{R^4} + \dots \tag{15}$$

It is better to express everything in terms of h to get

$$\log z(h) = \pi A h^{1/(1-\Delta_\sigma)} + \log(2^{c/3} z_\infty) - \frac{c}{6-6\Delta_\sigma} \log h + a_1 h^{-1/(1-\Delta_\sigma)} + \dots \tag{16}$$

where $z_\infty = \lambda^{c/(6-6\Delta_\sigma)} Z_\infty$ and $a_1 = 4b_1 \lambda^{1/(1-\Delta_\sigma)} \dots$

2.1. Vacuum expectation values

We will be interested also in one-point functions of relevant operators (VEV), which in flat space give essential information about short-distance expansion of correlation functions.

By expressing λ in terms of h and defining

$$G_{\Phi}(h) \equiv (2R)^{2\Delta_{\sigma}} \langle \Phi(0) \rangle_{\lambda} \quad (17)$$

one can show that

$$G_{\Phi}(h) = \sum_{n=1}^{\infty} g_n (-h)^n \quad (18)$$

where

$$g_n = \frac{1}{(2\pi)^n n!} \int \langle \Phi(0) \sigma(x_1) \cdots \sigma(x_n) \rangle_c \prod_{i=1}^n \frac{d^2 x_i}{(1 + x_i \bar{x}_i)^{2-2\Delta}} \quad (19)$$

and the subscript c means connected with respect to $\Phi(0)$.

Using (11) it is also possible to write a large R expansion of the form

$$G_{\Phi}(h) = A_{\Phi} h^{\frac{\Delta_{\Phi}}{1-\Delta_{\sigma}}} + a_1 h^{-\frac{\Delta_{\Phi}-1}{\Delta_{\sigma}-1}} + \dots \quad (20)$$

We recall that, in our normalizations, the VEV of a primary field on the plane is given by

$$\langle \Phi \rangle_{\lambda} = A_{\Phi} \left(\frac{\lambda}{2\pi} \right)^{\frac{\Delta_{\Phi}}{1-\Delta_{\sigma}}}. \quad (21)$$

There exists a simple relation between the VEV of the perturbing operator and the derivative of the partition function,

$$G_{\sigma}(h) = -2 \frac{z'(h)}{z(h)} = \frac{d}{dh} \log z(h) \quad (22)$$

and if $z(h)$ is an entire function the same is true for $z'(h)$ (the unnormalized VEV of the perturbing operator). It follows that, the asymptotic expansion for $G_{\sigma}(h)$ is given by

$$G_{\sigma}(h) = -\frac{2\pi A}{1-\Delta_{\sigma}} h^{\frac{\Delta_{\sigma}}{1-\Delta_{\sigma}}} + \frac{c}{3-3\Delta_{\sigma}} h^{-1} + \dots \quad (23)$$

As a result, in the $h \rightarrow \infty$ limit, the usual relation between A and the amplitude of the perturbing operator holds:

$$A_{\sigma} = \frac{A}{1-\Delta_{\sigma}}. \quad (24)$$

3. Numerical results

3.1. VEVs on the plane

The truncated conformal space (TCS) approach enables us to study numerically the behaviour of both the VEVs and the partition function of the model. The aim of this section is to give an estimate of the VEVs of the primary operators of the Ising model, i.e. magnetization and energy, in the limit of infinite plane by means of the asymptotic formulae of section 2.1. Since the model is defined on the sphere, in the large h limit we shall be able to recover the amplitudes A_{Φ} , with $\Phi \equiv \sigma, \epsilon$, obtained on the plane, whose value is exactly known [6].

Our strategy proceeds as follows: first, we fit the data obtained from the TCS technique by means of the expansion of section 2.1 at finite values of the truncation level N (we considered

$N = 10, 11, 12, 13, 14$) in an interval of the variable h ranging from 150 to 200 (this choice is motivated by the requirement to be in an asymptotic region where truncation artefacts are not present); second, we perform an extrapolation of the VEVs $A_\phi(N)$ to the limit $N \rightarrow \infty$ by means of the following (conjectured, see [4]) law:

$$A_\phi(N) = A_\phi(\infty) + A_\phi^1 N^{-x} + \dots \tag{25}$$

where the constant $A_\phi(\infty)$ is the extrapolated value of the amplitude A_ϕ . In order to get rid of possible systematic errors, the fitting procedure we used is the same as in [7]. In this way, we obtained the estimates for the amplitudes⁴ A_σ and A_ϵ :

$$A_\sigma = 1.27759(6) \quad A_\epsilon = 2.004(8) \tag{26}$$

which are in perfect agreement with both their theoretical values and the existing numerical estimates.

3.2. Analyticity properties of the partition function

Let us consider the case of pure imaginary values of the coupling h . One can see that the asymptotic behaviour of the partition function changes dramatically, becoming oscillatory and showing a well-defined pattern of zeros. Such zeros are approximatively located at (when the leading order in the asymptotic expansion is considered, see [2])

$$\begin{aligned} -ih_n^{(a)} &= \left[A\pi \sin\left(\frac{\pi}{2-2d}\right) \right]^{d-1} \left(\frac{\pi c}{12-12d} + \frac{\pi}{2} + n\pi \right)^{1-d} \\ &= \left[A \sin\left(\frac{8\pi}{15}\right) \right]^{-15/16} \left(n + \frac{49}{90} \right)^{15/16} \quad n \geq 0 \end{aligned} \tag{27}$$

where for the Ising model we have $c = 1/2, d = 1/16, A = 0.168564\dots$ (we used the same notation as in [2]). Hence, one could ask whether the hypothesis of [2] to consider the partition function as an entire function is compatible with numerical data also in the present case. Taking advantage of the TCS approach, we were able to compute numerically the partition function for the following values of the truncation level $N = 10, 11, 12$. Figure 1 shows such numerical results together with the plot of the asymptotic expansion (truncated at the leading order). Furthermore, one can compare the asymptotic location of zeros with the corresponding numerical estimates coming from the TCS approach. The result of such comparison is shown in table 1. It is interesting to check the validity of (27) against the exact sum rule [2]:

$$\sum_{n=0}^{\infty} \frac{1}{h_n^2} = \frac{1}{7} = 0.142857\dots \tag{28}$$

Plugging (27) in the previous expression, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(h_n^{(a)})^2} = 0.146549\dots \tag{29}$$

which is below 3% of accuracy with respect to the exact result.

The previous findings strongly suggest that the partition function can be considered as an entire function of the coupling h .

⁴ Their actual value is written using the standard normalization, see e.g. [6], where the factor 2π in (6) is absent. It is simply obtained by replacing $A_\phi \rightarrow A_\phi(2\pi)^{\frac{\Delta_\phi}{\Delta_\sigma-1}}$.

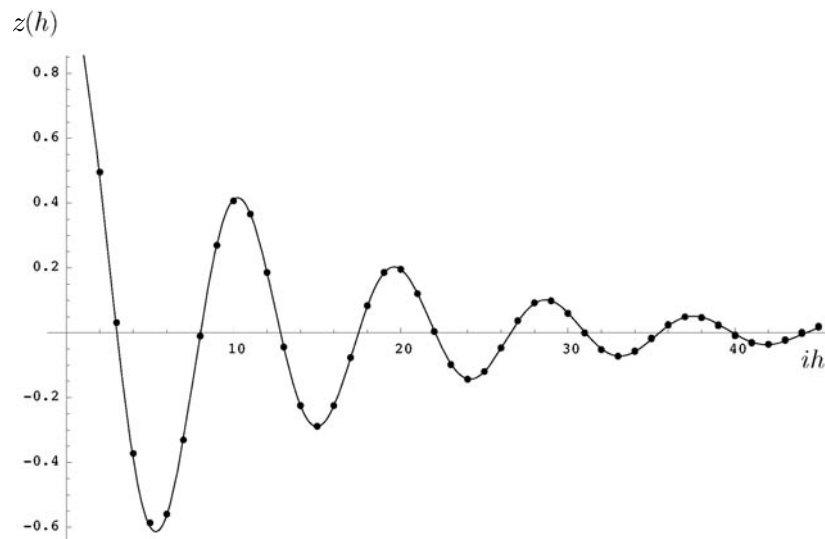


Figure 1. Behaviour of the partition function $z(ih)$. The solid line represents the asymptotic behaviour and dots show TCS approach results.

Table 1. The comparison between asymptotic formula (27) and TCS approach is shown at different values of the truncation level N . The agreement improves for higher values of h .

| Asymptotic | $N = 10$ | $N = 11$ | $N = 12$ |
|------------|----------|----------|----------|
| 3.017 17 | 3.068 82 | 3.068 86 | 3.068 88 |
| 8.018 96 | 8.031 76 | 8.032 33 | 8.032 36 |
| 12.805 2 | 12.804 3 | 12.805 8 | 12.806 4 |
| 17.472 1 | 17.460 3 | 17.463 1 | 17.464 3 |
| 22.056 3 | 22.029 6 | 22.033 4 | 22.035 3 |
| 26.577 3 | 26.537 8 | 26.545 7 | 26.548 5 |
| 31.047 4 | 30.980 7 | 31.000 7 | 31.005 3 |
| 35.474 8 | 35.368 4 | 35.415 3 | 35.423 1 |
| 39.865 5 | 39.686 5 | 39.776 3 | 39.790 3 |
| 44.224 2 | 43.940 6 | 44.089 5 | 44.113 4 |

As a final remark, we could ignore the exact knowledge of A and try to estimate it taking advantage of both the sum rules and formula (27). The numerical result

$$A_{\text{asy}} = \frac{1}{\sin \frac{8}{15}\pi} (7\zeta(15/8, 49/90))^{-8/15} = 0.1663 \dots$$

is remarkably near to the exact one $A = 0.168564 \dots$

4. Conclusions

Our results show that the partition function of the Ising model on the sphere may have interesting analytical properties in the coupling h .

The present case, together with the Yang–Lee model considered by Zamolodchikov in [2], seems to suggest the conjecture that all the perturbed rational conformal field theories (RCFTs) (when the perturbation is relevant) share similar analytic properties in their observables. In

particular, a proof of the convergence of the series defining the partition function should be welcome. An attempt in this direction could be tried along the same lines of the proof of convergence of strongly relevant ($2\Delta < 1$) perturbations of RCFTs in finite volume [10].

It would also be interesting to establish if there is a relation between the integrability of these models on the plane and their analytical properties on the sphere. In this perspective, some clarification could come from the study of a non-integrable perturbation⁵ of a given minimal model (e.g., the most relevant magnetic perturbation of the tricritical Ising model). Finally, we would like to stress that the VEV of primary (relevant) operators in the limit of infinite plane ($R \rightarrow \infty$) can be extracted with good precision using the TCS approach.

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⁵ Constantinescu and Flume [10] show that conformal perturbation theory is convergent in finite volume (which is similar to the present case) whether the theory is integrable or not. On the other hand, McCoy [8] pointed out that the results of Orrick *et al* [9] seem to suggest that the convergence of conformal perturbation theory fails if non-integrable perturbations are considered. A careful analysis of this apparent contradiction should be interesting, however it is beyond the purposes of this letter.